

THE JONES QUOTIENTS OF THE TEMPERLEY-LIEB ALGEBRAS.

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ABSTRACT. When the parameter q is a root of unity, the Temperley-Lieb algebra $\mathrm{TL}_n(q)$ is non-semisimple for almost all n . Jones showed that there is a canonical symmetric bilinear form on $\mathrm{TL}_n(q)$, whose radical $R_n(q)$ is generated by a certain idempotent $E_\ell \in \mathrm{TL}_{\ell-1}(q) \subseteq \mathrm{TL}_n(q)$, which is now referred to as the Jones-Wenzl idempotent, for which an explicit formula was subsequently given by Graham and Lehrer. In this work, we study the quotients $Q_n(\ell) := \mathrm{TL}_n(q)/R_n(q)$, where $|q^2| = \ell$, which are precisely the algebras generated by Jones' projections. We give the dimensions of their simple modules, as well as $\dim(Q_n(\ell))$; en route we give generating functions and recursions for the dimensions of cell modules and associated combinatorics. When the order $|q^2| = 4$, we obtain an isomorphism of $Q_n(\ell)$ with the even part of the Clifford algebra, well known to physicists through the Ising model. When $|q^2| = 5$, we obtain a sequence of algebras whose dimensions are the odd-indexed Fibonacci numbers. The general case is described explicitly.

1. INTRODUCTION

1.1. Preamble. In a celebrated series of works, Jones [8, 9, 10] showed that a certain sequence of von Neumann algebras A_n , $n = 1, 2, 3, \dots$ generated by idempotents $\{e_1, \dots, e_{n-1}\}$ in a type II_1 factor is closely related to the Temperley-Lieb algebras. His algebras depend on a parameter, denoted $q^2 \in \mathbb{C}$ in the present work, whose value in Jones' work was either in $\mathbb{R}_{\geq 0}$, or equal to $\exp(\pm \frac{2\pi i}{\ell})$ for $\ell = 3, 4, 5, \dots$. This constraint arose because of the context of the e_i as projections in a type II_1 factor (see [19]), which implies that a certain trace form on the algebra is positive definite. When $q^2 \in \mathbb{R}$, the projections e_i generate precisely the Temperley-Lieb algebra (see 1.1 below), and the positive definite nature of the trace form is automatic.

However the requirement that Jones' trace on A_n be non-degenerate implies that for $q^2 = \exp(\pm \frac{2\pi i}{\ell})$, the relevant algebra A_n is a proper quotient of the Temperley-Lieb algebra $\mathrm{TL}_n(q)$ (Definition 1.1 below) by the radical of the Jones form.

This radical of the trace form, regarded as a form on $\mathrm{TL}_n(q)$, is known [9, 7], [11, Thm. 2.1] to be generated by a single idempotent in $\mathrm{TL}_{\ell-1}(\exp(\pm \frac{2\pi i}{\ell}))$, known as the Jones-Wenzl idempotent. The main purpose of this work is to completely determine the dimension and representations of the quotient of $\mathrm{TL}_{\ell-1}(\exp(\pm \frac{2\pi i}{\ell}))$ by this radical, which we denote in this work by $Q_n(\ell)$ and which we refer to as the “Jones quotient”. When $\ell = 4$ we show below (Theorem 5.4) that, as is well known in the physics literature [12, 15], $Q_n(4)$ is isomorphic to the even part of the standard Clifford algebra. Our work could therefore be interpreted as the study of

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certain generalisations of the Clifford algebras. Other such generalisations exist in the literature (cf. [15, 16]), but are quite different from the Jones projection algebras $Q_n(\ell)$.

Our interest in this subject stemmed originally from a desire to understand how the Virasoro algebra arises as a limit of the Temperley-Lieb algebras, or of their semisimple quotients, which is a recurring theme in conformal field theory (see, for example, [12] and §6.5 below). We hope to return to this in the future.

1.2. The Temperley-Lieb algebras. In this work, all algebras will be over \mathbb{C} . Much of the theory we develop applies over more general domains, but since we will be concerned here with connections to the theory of operator algebras and mathematical physics, we limit our discussion to \mathbb{C} -algebras. For $n \in \mathbb{N}$, the Temperley-Lieb algebra $\text{TL}_n(q)$ is defined as follows.

Definition 1.1. Let $q \in \mathbb{C}$. $\text{TL}_n = \text{TL}_n(q)$ is the associative \mathbb{C} -algebra with generators f_1, f_2, \dots, f_{n-1} and relations

$$(1.1) \quad \begin{aligned} f_i^2 &= -(q + q^{-1})f_i \text{ for all } i \\ f_i f_{i\pm 1} f_i &= f_i \text{ for all } i \\ f_i f_j &= f_j f_i \text{ if } |i - j| \geq 2. \end{aligned}$$

In his seminal work [8] on subfactors of a factor, Jones showed that certain projectors $\{e_1, \dots, e_{n-1}\}$ ($n = 1, 2, 3, \dots$) in a von Neumann algebra satisfy the Temperley-Lieb-like relations, a fact that led to the definition of the ‘‘Jones polynomial’’ of an oriented link. In the notation of [10, p. 104, (I)–(VI)], Jones showed that if $f_i = (q + q^{-1})e_i$, then the f_i satisfy the relations (1.1), where Jones’ parameter t is replaced by q^2 . If $q^2 \neq -1$, Jones’ form on $\text{TL}_n(q)$ is defined as the unique (invariant) trace $\text{tr}_n = \text{tr}$ on $\text{TL}_n(q)$ which satisfies

$$(1.2) \quad \text{tr}(1) = 1 \text{ and } \text{tr}(x f_i) = -(q + q^{-1})^{-1} \text{tr}(x) \text{ for } x \in \text{TL}_{i-1} \subset \text{TL}_i \subseteq \text{TL}_n$$

for $1 \leq i \leq n - 1$.

This trace on $\text{TL}_n(q)$ is non-degenerate if and only if q^2 is not a root of unity, or, if $|q^2| = \ell$, $n \leq \ell - 2$ ([7, (3.8)]). Thus the discrete set of values of q^2 for which the Jones’ sequence (A_n) of algebras is infinite coincides precisely with the set of values of q^2 for which the trace form above on $\text{TL}_n(q)$ is degenerate. We are concerned in this work with the structure and representations of the quotient of $\text{TL}_n(q)$ by the radical of the Jones trace form.

1.3. Cell modules and forms. Let us fix n and consider the representation theory of TL_n . By [6] or [7], TL_n has cell modules $W_t := W_t(n)$ whose basis is the set of monic Temperley-Lieb morphisms from t to n , where $t \in \mathcal{T}(n)$, and $\mathcal{T}(n) = \{t \in \mathbb{Z} \mid 0 \leq t \leq n \text{ and } t + n \in 2\mathbb{Z}\}$.

Now W_t has an invariant form $(\ , \)$ which may be described as follows. For monic diagrams $D_1, D_2 : t \rightarrow n$, we form the diagram $D_1^* D_2 : t \rightarrow t$. If $D_1^* D_2$ is monic (i.e. a multiple of id_t), then we write $D_1^* D_2 = (D_1, D_2) \text{id}_t$; otherwise we say $(D_1, D_2) = 0$. Here D^* denotes the diagram obtained from D by reflection in a horizontal, extended to W_t by linearity.

2. SEMISIMPLICITY AND NON-DEGENERACY.

Clearly, if the trace (1.2) is non-degenerate, the algebra TL_n is semisimple. The converse is true except for one single case (see [7, Rem. 3.8, p.204]). It follows from [7, Cor. (3.6)] that if $|q^2| = \ell$, TL_n is non-semisimple if and only if $n \geq \ell$. Moreover we have very precise information concerning the radical of the invariant trace form (which we henceforth call the Jones form).

2.1. Radical of the trace form. The radical of the trace form above is given by the following result (see [7, §3], [9]).

Proposition 2.1. *If q is not a root of unity then tr is non-degenerate and TL_n is semisimple for all n .*

Suppose the order of q^2 is ℓ . Then there is a unique idempotent $E_{\ell-1} \in \text{TL}_{\ell-1}$ (the Jones-Wenzl idempotent) such that $f_i E_{\ell-1} = E_{\ell-1} f_i = 0$ for $1 \leq i \leq \ell - 2$. Moreover for $n \geq \ell$ the radical of tr_n is generated as ideal of TL_n by $E_{\ell-1}$.

Remark 2.2. cf. [7, Remark (3.8)] It follows from Proposition 2.1 that the trace tr_n is non-degenerate if and only if $n \leq \ell - 2$, where $\ell = |q^2|$. It follows that the case $n = \ell - 1$ is uniquely characterised as the one where the form tr is degenerate, but $\text{TL}_{\ell-1}$ is semisimple.

The following formula for the idempotent $E_{\ell-1}$ was proved in [7, Cor. 3.7]. To prepare for its statement, recall that if F is a finite forest (i.e. a partially ordered set in which $x \leq a, x \leq b \implies a \leq b$ or $b \leq a$), then we define a Laurent polynomial

$$(2.1) \quad h_F(x) = \frac{[|F|]_x!}{\prod_{a \in F} [F_{\leq a}]_x},$$

where, for $m \in \mathbb{N}$, $[m]_x = \frac{x^m - x^{-m}}{x - x^{-1}}$ and $[m]_x! = [m]_x [m-1]_x \cdots [2]_x [1]_x$.

Theorem 2.3. *For any Temperley-Lieb diagram $a : 0 \rightarrow 2n$ we have an associated forest F_a , which is simply the poset of arcs, ordered by their nesting. For any Temperley-Lieb diagram $D : t \rightarrow n$, one obtains a unique diagram $\overline{D} : 0 \rightarrow t + n$ by rotating the bottom line clockwise by π . With this notation, if $|q^2| = \ell$, we have*

$$(2.2) \quad E_{\ell-1} = \sum_D h_{F_{\overline{D}}}(q) D,$$

where the sum is over the diagrams from $\ell - 1$ to $\ell - 1$, i.e. over the diagram basis of $\text{TL}_{\ell-1}$.

Example 2.4. If $\ell = 4$, we may take $q = -\exp \frac{\pi i}{4}$, so that $q^2 = i$ and the element $E_3 \in \text{TL}_3$ is easily shown to be equal to

$$E_3 = 1 + f_1 f_2 + f_2 f_1 - \sqrt{2}(f_1 + f_2).$$

Note that our defining parameter for TL_n in this case is $-(q + q^{-1}) = \sqrt{2}$, and the above element is the familiar one which occurs in the study of the two-dimensional Ising lattice model.

2.2. The Jones quotient. We now wish to consider the quotient of TL_n by the ideal generated by $E_{\ell-1}$.

Definition 2.5. Assume that $|q^2| = \ell$ for a fixed integer $\ell \geq 3$. Let $J_\ell = \langle E_{\ell-1} \rangle$ be the ideal of $\mathrm{TL}_n(q)$ generated by the idempotent $E_{\ell-1} \in \mathrm{TL}_{\ell-1}(q)$, where $\mathrm{TL}_{\ell-1}(q)$ is thought of as a subalgebra of $\mathrm{TL}_n(q)$ for $n \geq \ell - 1$ in the obvious way.

The algebra $Q_n = Q_n(\ell)$ ($n = \ell - 1, \ell, \ell + 1, \dots$) is defined by

$$Q_n(\ell) = \frac{\mathrm{TL}_n}{J_\ell}.$$

This algebra will sometimes be referred to as the “Jones projection algebra”.

Since we are taking the quotient by the radical of the trace form tr_n , it follows that Q_n has a non-degenerate invariant trace, and hence that

$$(2.3) \quad Q_n \text{ is semisimple.}$$

Remark 2.6. We remark that $E_2 = 1 - f_1$, from which it follows that $Q_n(3) \cong \mathbb{C}$ for all n . We therefore henceforth assume that $\ell = 4, 5, 6, \dots$

3. REPRESENTATION THEORY OF Q_n .

We wish to understand the simple Q_n -modules, and ultimately, a “fusion rule”, which describes their tensor products. This will be approached through the cellular theory, which applies to the algebra TL_n .

3.1. Review of the representation theory of TL_n at a root of unity. Let $|q^2| = \ell$. As noted in Remark 2.6 it suffices to consider $\ell \geq 4$; further, since $Q_n(\ell) = \mathrm{TL}_n(q)$ for $n < \ell - 1$, we generally assume that $n \geq \ell - 1$.

Since $\mathrm{TL}_n(q)$ is cellular, it has cell modules $W_t(n)$, $t \in \mathcal{T}(n)$ (cf. §1.3). These have basis the set of monic diagrams $t \rightarrow n$, are generically simple, but occasionally have two composition factors. As mentioned above, $W_t(n)$ has an invariant form $(\ , \)_t$; let $L_t := W_t / \mathrm{Rad}_t$, where Rad_t is the radical of the form $(\ , \)_t$. Then the general theory asserts that the L_t are simple, and represent all the distinct isomorphism classes of simple $\mathrm{TL}_n(q)$ -modules.

The following description of the composition factors of $W_t = W_t(n)$ was given in [7, Thm. 5.3], and in the formulation here in [1, Thm. 6.9].

Theorem 3.1. *Let $|q^2| = \ell$, fix $n \geq \ell$ and let $\mathcal{T}(n)$ be as above. Let $\mathbb{N}' = \{i \in \mathbb{N} \mid i \not\equiv -1 \pmod{\ell}\}$. Define $g : \mathbb{N}' \rightarrow \mathbb{N}'$ as follows: for $t = a\ell + b \in \mathbb{N}'$, $0 \leq b \leq \ell - 2$, define $g(t) = (a + 1)\ell + \ell - 2 - b$. Notice that $g(t) - t = 2(\ell - b - 1)$, so that $g(t) \geq t + 2$ and $g(t) \equiv t \pmod{2}$.*

- (1) *For $t \in \mathcal{T}(n) \cap \mathbb{N}'$ such that $g(t) \in \mathcal{T}(n)$, there is a non-zero homomorphism $\theta_t : W_{g(t)}(n) \rightarrow W_t(n)$. These are explicitly described in [7, Thm 5.3], and are the only non-trivial homomorphisms between the cell modules of TL_n .*
- (2) *If $t \in \mathcal{T}(n)$ is such that $t \in \mathbb{N}'$ and $g(t) \in \mathcal{T}(n)$, then $W_t(n)$ has composition factors L_t and $L_{g(t)}$, each with multiplicity one. All other cell modules for $\mathrm{TL}_n(q)$ are simple.*
- (3) *If $\ell \geq 3$, all the modules L_t , $t \in \mathcal{T}(n)$, are non-zero, and form a complete set of simple $\mathrm{TL}_n(q)$ -modules.*

3.2. Representation theory of Q_n . We have seen (2.3) that the algebras Q_n are semisimple. We therefore focus on the description of their simple modules.

Proposition 3.2. *Let $n \geq \ell - 1$. The simple Q_n -modules are precisely those simple TL_n -modules L_t , $t \in \mathcal{T}(n)$, such that $J_\ell L_t = 0$, where J_ℓ is the ideal of TL_n generated by $E_{\ell-1}$.*

Proof. Recall that $Q_n = \text{TL}_n / J_\ell$. If M is any Q_n -module, it may be lifted via the canonical surjection $\text{TL}_n \xrightarrow{\eta_n} Q_n$ to a TL_n -module which we denote by \widetilde{M} , on which J_ℓ acts trivially. Conversely, if \widetilde{M} is any TL_n -module on which J_ℓ acts trivially, the action factors through $\text{TL}_n / J_\ell = Q_n$, so that \widetilde{M} may be thought of as a Q_n -module M .

Moreover, it is clear that M is simple as Q_n -module if and only if \widetilde{M} is simple as TL_n -module.

Suppose now that M is a simple Q_n -module. Then \widetilde{M} is a simple TL_n -module, and hence by Theorem 3.1 (3), is isomorphic to L_t for some $t \in \mathcal{T}(n)$ and by the above remarks, J_ℓ acts trivially on L_t . Conversely, if L_t satisfies $J_\ell L_t = 0$, then L_t is a simple Q_n -module. \square

Remark 3.3. If N is a TL_n module, then since $J_\ell = \text{TL}_n E_{\ell-1} \text{TL}_n$, it follows that $J_\ell N = 0$ if and only if $E_{\ell-1} N = 0$. Thus the condition in the Proposition is relatively straightforward to check.

Remark 3.4 (Remark concerning notation). Although *a priori* $E_{\ell-1} \in \text{TL}_{\ell-1}$, we have regarded it as an element of TL_n for any $n \geq \ell - 1$. The strictly correct notation for $E_{\ell-1} \in \text{TL}_n$, where $n \geq \ell$, is $E_{\ell-1} \otimes I^{\otimes(n-\ell+1)}$, where the tensor product is in the Temperley-Lieb category \mathbf{T} , as described in [7] or [14]; that is, it is described diagrammatically as juxtaposition of diagrams, and I is the identity diagram from 1 to 1. We shall use this notation freely below.

Theorem 3.5. *With notation as in Theorem 3.1, let $t \in \mathcal{T}(n)$ satisfy $t \geq \ell - 1$. Then the idempotent $E_{\ell-1} \otimes I^{\otimes(n-\ell+1)}$ acts non-trivially on L_t . Thus Q_n has at most $\lfloor \frac{\ell}{2} \rfloor$ isomorphism classes of simple modules.*

Proof. We begin by showing that if $t \in \mathcal{T}(n)$ and $t \geq \ell - 1$ then $E_{\ell-1} \otimes I^{\otimes(n-\ell+1)} W_t$ contains all diagrams of the form $I^{\otimes t} \otimes D'$, where D' is any monic diagram from 0 to $n - t$.

To see this, note that W_t is spanned by monic diagrams from t to n in \mathbf{T} . Take $D = I^{\otimes t} \otimes D' \in W_t$, where D' is any (monic) diagram from 0 to $n - t$. By the formula in Theorem 2.3, the coefficient of $I^{\otimes(\ell-1)}$ in $E_{\ell-1}$ is 1. Since all the other summands act trivially on D (because they reduce the number of ‘through strings’), it follows that $E_{\ell-1} \otimes I^{\otimes(n-\ell+1)} D = D$ in W_t , and hence that $D \in E_{\ell-1} \otimes I^{\otimes(n-\ell+1)} W_t$.

Now if $D = I^{\otimes t} \otimes D'$ as above and ϕ_t is the canonical bilinear form on W_t (see [6, §2]), then $\phi_t(D, D)$ is a power of $-(q + q^{-1})$, and hence is non-zero. It follows from [6, Cor. (2.5)(ii)] that $W_t = \text{TL}_n D$, whence $W_t = J_\ell W_t$. Moreover it also follows that $D \notin \text{Rad}_t$, and hence that modulo Rad_t , D generates L_t , whence $J_\ell L_t = L_t$. \square

It follows from the above result that the only possible simple Q_n -modules are the L_t with $t < \ell - 1$.

Theorem 3.6. *The simple Q_n modules are the L_t with $t \leq \ell - 2$.*

Proof. In view of Proposition 3.2 and Theorem 3.5, it suffices to show that $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})L_{\ell-2} = 0$ and $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})W_t = 0$ for $t \leq \ell - 3$.

Consider first the case $t \leq \ell - 3$. Then any monic diagram $D : t \rightarrow n$ contains an upper horizontal arc whose right vertex $\leq \ell - 1$. Since $E_{\ell-1}$ is harmonic in $\text{TL}_{\ell-1}$, it follows that $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D = 0$, and hence that $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})W_t = 0$.

Now consider the case $t = \ell - 2$. Then $g(t) = \ell$ (see the statement of Theorem 3.1) and it follows from Theorem 3.1(2) and the fact that $n \geq \ell$, that $W_{\ell-2}(n)$ has composition factors $L_{\ell-2}$ and L_ℓ . If $D : t \rightarrow n$ is a monic diagram, then by the harmonic nature of $E_{\ell-1}$, $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D = 0$ unless $D = I^{\otimes(\ell-2)} \otimes D'$, where D' is a diagram from 0 to $n - \ell + 2$. But in this case an inspection of the diagrams shows that if $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D = x \in W_{\ell-2}$, then $\phi_{\ell-2}(x, x)$ is a multiple of $\text{tr}_{\ell-1}(E_{\ell-1}) = 0$.

More generally, if D_1, D_2 are diagrams in $W_{\ell-2}$ and $(E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D_i = x_i$ ($i = 1, 2$), then the same argument shows that $\phi_{\ell-2}(x_1, x_2) = 0$.

It follows that for any diagrams $D_1, D_2 \in W_{\ell-2}$, $\phi_{\ell-2}((E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D_1, D_2) = 0$, since $E_{\ell-1} \otimes I^{\otimes(n-\ell+1)}$ is idempotent and self dual, so that

$$\begin{aligned} \phi_{\ell-2}((E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D_1, D_2) &= \phi_{\ell-2}((E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})^2 D_1, D_2) \\ &= \phi_{\ell-2}((E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D_1, (E_{\ell-1} \otimes I^{\otimes(n-\ell+1)})D_2) \\ &= 0. \end{aligned}$$

Hence $J_\ell W_{\ell-2} \subseteq \text{Rad}_{\ell-2}$, and it follows that $J_\ell L_{\ell-2} = 0$. \square

4. DIMENSIONS.

In this section we shall discuss the dimensions of the simple $Q_n(\ell)$ -modules. To begin with, we give explicit formulae for the dimensions of the cell modules of TL_n .

4.1. Cell modules for TL_n . Recall that the cell module $W_t(n)$ has a basis consisting of the monic TL-diagrams $D : t \rightarrow n$. Since such diagrams exist only when $t \equiv n \pmod{2}$, we may write $n = t + 2k$, $k \geq 0$.

Definition 4.1. For $t, k \geq 0$, we write $w(t, k) := \dim W_t(t + 2k)$. By convention, $W_0(0) = 0$, so that $w(0, 0) = 0$.

Proposition 4.2. *We have the following recursion for $w(t, k)$. For integers $t, k \geq 0$:*

$$(4.1) \quad w(t, k + 1) = w(t - 1, k + 1) + w(t + 1, k).$$

Proof. The proof is based on the interpretation of $w(t, k)$ as the number of monic TL-diagrams from t to $t + 2k$.

Consider first the case $t = 0$. The assertion is then that $w(0, k + 1) = w(1, k)$. But all \mathbf{T} -diagrams $D : 1 \rightarrow 1 + 2k$ are monic, as are all diagrams $0 \rightarrow 2\ell$ (any ℓ). It follows that $w(1, k) = \dim(\text{Hom}_{\mathbf{T}}(1, 1 + 2k)) = \dim(\text{Hom}_{\mathbf{T}}(0, 2 + 2k)) = w(0, k + 1)$. Thus the assertion is true for $t = 0$ and all $k \geq 0$. Similarly, if $k = 0$, the assertion amounts to $w(t, 1) = w(t - 1, 1) + w(t + 1, 0)$. If $t > 0$, the left side is easily seen to be equal to $t + 1$, while $w(t - 1, 1) = t$ and $w(t + 1, 0) = 1$. If $t = 0$, the left side is equal to $\dim(\text{Hom}_{\mathbf{T}}(0, 2k + 2)) = \dim(\text{Hom}_{\mathbf{T}}(1, 2k + 1)) = w(1, k)$. So the recurrence is valid for $k = 0$ and all t .

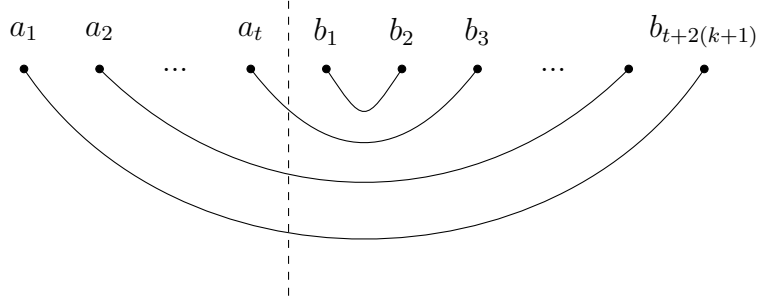


FIGURE 1. Monic diagram $t \rightarrow t + 2(k + 1)$ as a diagram $0 \rightarrow 2t + 2(k + 1)$.

Now consider the general case. Our argument will use the fact that $w(t, k + 1)$ may be thought of as the number of TL-diagrams $0 \rightarrow 2t + 2(k + 1)$ of the form depicted in Fig. 1.

The condition that the diagram be monic is simply that each a_i is joined to some b_j , i.e. that each arc crosses the dotted line; of course distinct arcs are non-intersecting.

Now evidently such diagrams fall into two types: those in which $[a_t, b_1]$ is an arc, and the others. Now the number of diagrams in which $[a_t, b_1]$ is an arc is clearly equal to $w(t - 1, k + 1)$, while those in which $[a_t, b_1]$ is not an arc are in bijection with the monic diagrams from $t + 1$ to $t + 1 + 2k$, as is seen by shifting the dotted line one unit to the right. Hence the number of the latter is $w(t + 1, k)$, and the recurrence (4.1) is proved. \square

4.2. A binomial expression for $w(t, k)$.

Definition 4.3. For integers $t, k \geq 0$, define

$$(4.2) \quad F(t, k) = \binom{t + 2k}{k} - \binom{t + 2k}{k - 1}.$$

This definition is extended to the domain $\mathbb{Z} \times \mathbb{Z}$ by stipulating that $F(t, k) = 0$ if $t < 0$ or $k < 0$.

It is easily seen that

$$(4.3) \quad F(t, k) = \frac{(t + 1)(t + 2k)(t + 2k - 1) \dots (t + k + 2)}{k!} = \frac{t + 1}{t + k + 1} \binom{t + 2k}{k}.$$

Lemma 4.4. We have the following recursion for $F(t, k)$. For $t, k \geq 0$:

$$(4.4) \quad F(t, k + 1) = F(t - 1, k + 1) + F(t + 1, k).$$

Proof. For any integers $r, n \geq 0$ we have the binomial identity $\binom{n}{r+1} = \binom{n-1}{r+1} + \binom{n-1}{r}$. Using this in the form $\binom{n}{r+1} - \binom{n-1}{r+1} = \binom{n-1}{r}$, we see from (4.2) that

$$F(t, k + 1) - F(t - 1, k + 1) = \binom{t + 2k + 1}{k} - \binom{t + 2k + 1}{k - 1} = F(t + 1, k).$$

\square

4.3. Catalan calculus–generating functions. For $n \geq 0$, write $c(n) := w(0, 2n)$, $c(0) = 1$. It is easily seen by inspecting diagrams that for $n > 0$,

$$(4.5) \quad c(n) = \sum_{k=1}^n c(k-1)c(n-k).$$

Writing $c(x) := \sum_{n=0}^{\infty} c(n)x^n$, the recursion (4.5) translates into

$$(4.6) \quad xc(x)^2 - c(x) + 1 = 0,$$

from which it is immediate that

$$(4.7) \quad c(x) = \frac{1 - (1 - 4x)^{\frac{1}{2}}}{2x},$$

and applying the binomial expansion, that

$$(4.8) \quad c(n) = \frac{1}{n+1} \binom{2n}{n}.$$

Now define $W_t(x) = \sum_{k=0}^{\infty} w(t, k)x^k$. Inspection of diagrams shows that the $w(t, k)$ satisfy the following recursion.

$$(4.9) \quad w(t, k) = \sum_{\ell=0}^k w(t-1, \ell)c(k-\ell),$$

which translates into the recursion $W_t(x) = W_{t-1}(x)c(x)$ for the generating function $W_t(x)$. Using the fact that evidently $W_0(x) = c(x)$, we have proved the following statement.

Proposition 4.5. *For $t = 0, 1, 2, \dots$, we have $W_t(x) = \sum_{k=0}^{\infty} w(t, k)x^k = c(x)^{t+1}$.*

Corollary 4.6. *We have the following equation in $\mathbb{Z}[[x, y]]$.*

$$(4.10) \quad W(x, y) := \sum_{t, k=0}^{\infty} w(t, k)y^t x^k = \frac{c(x)}{1 - yc(x)}.$$

4.4. A closed expression for $w(t, k)$. We shall prove the following theorem.

Theorem 4.7. *For integers $t, k \geq 0$, we have*

$$(4.11) \quad w(t, k) = F(t, k).$$

That is,

$$(4.12) \quad \dim(W_t(t+2k)) = \frac{t+1}{t+k+1} \binom{t+2k}{k}.$$

Proof. Consider first the case $k = 0$. Then for any $t \geq 0$, $w(t, 0) = 1$, while $F(t, 0) = \frac{1}{1} \binom{t}{0} = 1$. Thus the result is true for $k = 0$ and all t . Next consider the case $t = 0$. Then for all $k \geq 0$, $w(0, k) = c(k) = \frac{1}{k+1} \binom{2k}{k}$, while $w(0, k)$ is evidently also equal to $c(k)$ by (4.3). Thus the assertion is true for $t = 0$ and all k .

Now suppose that the following assertion is true.

P(\mathbf{t}_0): *The result is true for pairs (t, k) such that $t \leq t_0$ (and any $k \geq 0$).*

We now use the recursions (4.1) and (4.4), in the following form.

$$(4.13) \quad \begin{aligned} F(t_0 + 1, k) &= F(t_0, k + 1) - F(t_0 - 1, k + 1) \text{ and} \\ w(t_0 + 1, k) &= w(t_0, k + 1) - w(t_0 - 1, k + 1). \end{aligned}$$

By assumption, the right sides of (4.13) are equal, whence $F(t_0 + 1, k) = w(t_0 + 1, k)$, i.e. $\mathbf{P}(t_0 + 1)$ holds. Since we have seen that $\mathbf{P}(0)$ is true, the result follows. \square

4.5. Application to the simple modules for $Q_n(\ell)$. Fix $\ell \geq 3$ and assume that q^2 has order ℓ . We wish to study the dimensions of the simple Q_n -modules, which we have already seen (Theorem 3.6) are the simple TL_n -modules $L_t(n)$ for $0 \leq t \leq \ell - 2$, and $t \equiv n \pmod{2}$.

Definition 4.8. It will be convenient to denote the dimensions of the cell modules $W_t(n)$ and simple modules $L_t(n)$ for the algebra TL_n by $w_t(n)$ and $\ell_t(n)$ respectively. We regard w_t and ℓ_t as functions of $n \in \mathbb{Z}_{\geq 0}$. These functions satisfy the following conditions: $w_t(n) = 0$ if $t > n$ or if $t < 0$. More generally, $w_t(n) = \ell_t(n) = 0$ for any $t \in \mathbb{Z}$ such that $t \notin \mathcal{T}(n)$ (see Theorem 3.1).

We have seen by Theorem 3.1 that in general, the composition factors of the cell module $W_t(n)$ (for $n \in \mathbb{N}'$) are $L_t(n)$ and $L_{g(t)}(n)$ where $g : \mathbb{N}' \rightarrow \mathbb{N}'$ is the function defined in Theorem 3.1. The following result is an easy consequence of this.

Proposition 4.9. *Let $|q^2| = \ell$. We have, for $t \in \mathbb{N}' = \{s \in \mathbb{Z}_{\geq 0} \mid s \not\equiv -1 \pmod{\ell}\}$:*

$$(4.14) \quad \ell_t(n) = \sum_{i=0}^{\infty} (-1)^i w_{g^i(t)}(n).$$

Proof. Note that since g is a strictly increasing function on \mathbb{N}' , for any particular n , the sum on the right side of (4.14) is finite.

It is evident from Theorem 3.1 (2), that for any $t \in \mathbb{N}'$,

$$(4.15) \quad \ell_t(n) = w_t(n) - \ell_{g(t)}(n).$$

Applying (4.15) with t replaced by $g(t)$ gives $\ell_t(n) = w_t(n) - w_{g(t)}(n) + \ell_{g^2(t)}(n)$. Applying this repeatedly, and noting that there is an integer $t_0 \in \mathbb{N}'$ such that $t_0 \leq n$ and $g(t_0) > n$, we obtain the relation (4.14). \square

Remark 4.10. (1) In view of the interpretation of w_t and ℓ_t as functions (see Definition 4.8), the relation (4.14) will be written as the following equality of functions on \mathbb{N}' .

$$(4.16) \quad \ell_t = \sum_{i=0}^{\infty} (-1)^i w_{g^i(t)}.$$

(2) Note that by Theorem 3.6, the simple Q_n -modules are those L_t with $0 \leq t \leq \ell - 2$. Thus for these modules, $t \in \mathbb{N}'$, and the analysis above applies to all the simple modules.

5. THE ISING ALGEBRA $Q_n(4)$

We shall now give a detailed analysis of the particular case $\ell = 4$. We refer to Q_n in this case as the Ising algebra. In this section, $\ell = 4$, so that $q^2 = \sqrt{-1}$ and we take $q = -\exp \frac{\pi i}{4}$. The first line of the relation (1.1) then reads:

$$f_i^2 = \sqrt{2}f_i.$$

5.1. The function g . The function $g(t)$ ($t \in \mathbb{N}$, $t \not\equiv -1 \pmod{4}$) defined in Theorem 3.1 is easily described in this case. We have

$$(5.1) \quad g(t) = \begin{cases} t+6 & \text{if } t \equiv 0 \pmod{4} \\ t+2 & \text{if } t \equiv 2 \pmod{4} \\ t+4 & \text{if } t \equiv 1 \pmod{4}. \end{cases}$$

5.2. Structure and dimension of Q_n . The main theorem of this section is

Theorem 5.1. *Let $\ell = |q^2| = 4$ and for $n = 3, 4, 5, 6, \dots$, let Q_n be the Ising algebra defined in Definition 2.5. The dimensions of the simple Q_n -modules are as follows. If n is even, then Q_n has just two simple modules L_0 and L_2 , whose dimensions $\ell_0(n)$ and $\ell_2(n)$ are given by*

$$(5.2) \quad \ell_0(n) = \ell_2(n) = 2^{\frac{n}{2}-1}.$$

If n is odd, then Q_n has a unique simple module $L_1(n)$, whose dimension $\ell_1(n)$ is given by

$$(5.3) \quad \ell_1(n) = 2^{\frac{n-1}{2}}.$$

Proof. Combining the general formula (4.14) with the particular values (5.1) of the function g in this case, we obtain the following expressions for the relevant dimensions.

$$(5.4) \quad \begin{aligned} \ell_0 &= \sum_{t \in \mathbb{N}, t \equiv 0 \pmod{8}} w_t - \sum_{t \in \mathbb{N}, t \equiv 6 \pmod{8}} w_t, \\ \ell_2 &= \sum_{t \in \mathbb{N}, t \equiv 2 \pmod{8}} w_t - \sum_{t \in \mathbb{N}, t \equiv 4 \pmod{8}} w_t \text{ and} \\ \ell_1 &= \sum_{t \in \mathbb{N}, t \equiv 1 \pmod{8}} w_t - \sum_{t \in \mathbb{N}, t \equiv 5 \pmod{8}} w_t. \end{aligned}$$

We shall prove the relations (5.2) and (5.3) by induction on n . Observe that if $n = 4$, $\ell_0(4) = w_0(4) = 2$, and $\ell_2(4) = w_2(4) - w_4(4) = 3 - 1 = 2$, so that (5.2) is true for $n = 4$. Similarly $\ell_1(3) = w_1(3) = 2$, so that (5.3) holds for $n = 3$; moreover $\ell_1(5) = w_1(5) - w_5(5) = 5 - 1 - 4$, so that (5.3) also holds for $n = 5$.

Next, notice that the recursion (4.1) for $w_t(n)$ may be written

$$(5.5) \quad w_t(n+1) = w_{t-1}(n) + w_{t+1}(n) \text{ for } t \geq 0 \text{ and } n \in \mathbb{N}'.$$

Applying the same recursion to both terms on the right side of (5.5) and replacing n by $n+1$, we obtain

$$(5.6) \quad w_t(n+2) = \begin{cases} w_{t-2}(n) + 2w_t(n) + w_{t+2}(n) & \text{if } t \geq 1 \\ w_t(n) + w_{t+2}(n) & \text{if } t = 0. \end{cases}$$

Note that the relation (5.6) holds for all $n \in \mathbb{Z}$ and t in the stated range, subject to the interpretation of w_t given in Definition 4.8.

Let us now compute $\ell_1(n)$. Since the evaluation of the infinite sums in (5.4) at any $n \in \mathbb{N}'$ involves only a finite sum, we may rearrange the terms in any way we choose. The third line of (5.4) may therefore be written as follows.

$$(5.7) \quad \ell_1 = \sum_{t \in \mathbb{N}, t \equiv 1 \pmod{8}} (w_t - w_{t+4}).$$

But it follows from two applications of the first line of (5.6) that for any odd integer $n \geq 3$ and $t \geq 1$, we have

$$(5.8) \quad (w_t - w_{t+4})(n+2) = (w_{t-2} - w_{t+6})(n) + 2(w_t - w_{t+4})(n).$$

Summing (5.8) over $t \geq 0$ such that $t \equiv 1 \pmod{8}$, we obtain, using (5.7),

$$\ell_1(n+2) = \sum_{t \geq 1, t \equiv 1 \pmod{8}} (w_{t-2}(n) - w_{t+6}(n)) + 2\ell_1(n).$$

Moreover $\sum_{t \geq 1, t \equiv 1 \pmod{8}} (w_{t-2}(n) - w_{t+6}(n)) = w_{-1}(n) + w_7(n) - w_7(n) + w_{15}(n) - w_{15}(n) + \dots = 0$. Hence $\ell_1(n+2) = 2\ell_1(n)$ for $n \geq 3$, and since $\ell_0(3) = 2$, it follows by induction that $\ell_1(2r+1) = 2^r$ ($r \geq 1$), as asserted in (5.3).

Next consider $\ell_2(n)$ for $n \geq 4$ even. The second line of (5.4) may be written

$$(5.9) \quad \ell_2 = \sum_{t \in \mathbb{N}, t \equiv 2 \pmod{8}} (w_t - w_{t+2}).$$

A similar argument to that above shows that for $t \geq 2$,

$$(5.10) \quad (w_t - w_{t+2})(n+2) = (w_{t-2} - w_{t+4})(n) + (w_t - w_{t+2})(n),$$

and summing over t such that $t \equiv 2 \pmod{8}$, we see that

$$(5.11) \quad \ell_2(n+2) = \ell_0(n) + \ell_2(n).$$

Now consider ℓ_0 . In this case, we have

$$(5.12) \quad \ell_0 = \sum_{t \in \mathbb{N}, t \equiv 0 \pmod{8}} (w_t - w_{t+6}).$$

In view of (5.6) we need to treat the case $t = 0$ separately. Arguing as above, we obtain

$$(5.13) \quad (w_t - w_{t+6})(n+2) = \begin{cases} (w_{t-2} - w_{t+6})(n) + (w_t - w_{t+8})(n) + (w_t - w_{t+6})(n) \\ \quad + (w_{t+2} - w_{t+4})(n) \text{ if } t \geq 2, \\ (w_0 + w_2 - w_4 - 2w_6 - w_8)(n) \text{ if } t = 0. \end{cases}$$

One now performs the straightforward but detailed task of computing the sum $\sum_{t \in \mathbb{N}, t \equiv 0 \pmod{8}} (w_t - w_{t+6})(n+2)$ using (5.13). This shows that

$$(5.14) \quad \ell_0(n+2) = \ell_0(n) + \ell_2(n).$$

Since $\ell_0(4) = \ell_2(4) = 2$, the relations (5.11) and (5.14) complete the proof of the Theorem by induction. \square

Corollary 5.2. *Let $\ell = |q^2| = 4$. The dimension of the Ising algebra Q_n defined in Definition 2.5 is 2^{n-1} .*

Proof. We have seen that Q_n is a semisimple finite dimensional algebra. Its dimension is therefore the sum of the squares of the dimensions of its distinct simple modules. The statement now evidently follows from Theorem 5.1. \square

5.3. The Clifford algebra and Q_n . Let U be a complex vector space of finite dimension n , with a non-degenerate symmetric bilinear form $\langle -, - \rangle$. Then U has an orthonormal basis u_1, \dots, u_n , which enjoys the property that $\langle u_i, u_j \rangle = \delta_{ij}$. If $\gamma_i = \frac{1}{\sqrt{2}}u_i$ for $i = 1, \dots, n$, then for any i, j ,

$$(5.15) \quad \langle \gamma_i, \gamma_j \rangle = \frac{1}{2}\delta_{i,j}.$$

The Clifford algebra $\mathcal{C}_n = \mathcal{C}(U, \langle -, - \rangle)$ (for generalities about Clifford algebras we refer the reader to [5]) is defined as

$$(5.16) \quad \mathcal{C}_n = \frac{T(U)}{I},$$

where $T(U) = \bigoplus_{i=0}^{\infty} U^{\otimes i}$ is the free associative \mathbb{C} -algebra (or tensor algebra) on U , and I is the ideal of $T(U)$ generated by all elements of the form $u \otimes u - \langle u, u \rangle 1$ ($u \in U$). This last relation may equivalently be written (omitting the \otimes in the multiplication)

$$(5.17) \quad uv + vu = 2\langle u, v \rangle 1.$$

The algebra \mathcal{C}_n is evidently generated by any basis of U , and hence by (5.15) and (5.17) has the presentation

$$(5.18) \quad \mathcal{C}_n = \langle \gamma_1, \dots, \gamma_n \mid \gamma_i \gamma_j + \gamma_j \gamma_i = \delta_{ij} \text{ for } 1 \leq i, j \leq n \rangle.$$

For any subset $J = \{j_1 < j_2 < \dots < j_p\} \subseteq \{1, \dots, n\}$, write $\gamma_J = \gamma_{j_1} \gamma_{j_2} \dots \gamma_{j_p}$. It is evident that $\{\gamma_J \mid J \subseteq \{1, 2, \dots, n\}\}$ is a basis of \mathcal{C}_n which is therefore \mathbb{Z}_2 -graded (since the relations are in the even subalgebra of the tensor algebra), the even (resp. odd) subspace being spanned by those γ_J with $|J|$ even (resp. odd).

The following statement is now clear.

Proposition 5.3. *The Clifford algebra $\mathcal{C}(U, \langle -, - \rangle)$ has dimension 2^n , where $n = \dim(U)$. Its even subalgebra \mathcal{C}_n^0 has dimension 2^{n-1} .*

The next theorem is the main result of this section; it asserts that the Ising algebra is isomorphic to the even subalgebra of the Clifford algebra.

Theorem 5.4. *We continue to assume $\ell = 4$ and that $q = -\exp(\frac{\pi i}{4})$. Other notation is as above. For $n = 3, 4, \dots$ there are surjective homomorphisms $\phi_n : \text{TL}_n(q) \rightarrow \mathcal{C}_n^0$ which induce isomorphisms $\bar{\phi}_n : Q_n \xrightarrow{\sim} \mathcal{C}_n^0$.*

Proof. Define $\phi_n(f_j) = \frac{1}{\sqrt{2}}(1 + 2i\gamma_j\gamma_{j+1})$. It was remarked by Koo and Saleur [12, §3.1 eq. (3.2)] (see also [2]) that the $\phi_n(f_j)$ satisfy the relations (1.1) in \mathcal{C}_n , and therefore that ϕ_n defines a homomorphism from TL_n to \mathcal{C}_n , and further that $E_3 \in \ker(\phi_n)$.

It is evident that the image of ϕ_n is \mathcal{C}_n^0 , and therefore that $\bar{\phi}_n : Q_n \rightarrow \mathcal{C}_n^0$ is surjective. But by Cor. 5.2 and Prop. 5.3 these two algebras have the same dimension, whence $\bar{\phi}_n$ is an isomorphism. \square

5.4. Canonical trace. Let $\text{TL}_n(q)$ be the n -string Temperley-Lieb algebra as above, and assume $\delta := -(q + q^{-1}) \neq 0$ is invertible. The canonical Jones trace tr_n on $\text{TL}_n(q)$ is characterised by the properties

$$(5.19) \quad \begin{aligned} \text{tr}_n(1) &= 1, \\ \text{tr}_{n+1}(xf_n) &= \delta^{-1} \text{tr}_n(x) \text{ for } x \in \text{TL}_n(q). \end{aligned}$$

As has been pointed out above (2.3), this trace descends to a non-degenerate trace on Q_n , satisfying similar properties.

Proposition 5.5. *There is a canonical trace $\overline{\text{tr}}_n$ on \mathcal{C}_n^0 , given by taking the constant term (coefficient of 1) of any of its elements. This trace corresponds to the Jones trace above in the sense that for $x \in Q_n$, $\text{tr}_n(x) = \overline{\text{tr}}_n(\phi(x))$. It is therefore non-degenerate.*

The proof is easy, and consists in showing that $\overline{\text{tr}}_n$ satisfies the analogue of (5.19) in \mathcal{C}_n^0 .

5.5. The spinor representations of $\mathfrak{so}(n)$. We give yet another interpretation of the algebra in terms of the spin representations of $\mathfrak{so}(n)$. Let $\text{SO}(n)$ be the special orthogonal group of the space $(U, \langle -, - \rangle)$ above. Its Lie algebra has basis the set of matrices (with respect to the orthogonal basis (γ_i)) $J_{ij} := E_{ij} - E_{ji}$, $1 \leq i < j \leq n$, where the E_{ij} are the usual matrix units. This basis of $\mathfrak{so}(n)$ satisfies the commutation relations

$$(5.20) \quad [J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{jl} J_{ik} - \delta_{ik} J_{jl} + \delta_{il} J_{jk}.$$

Proposition 5.6. *For $n \geq 2$, there are surjective homomorphisms $\psi_n : \text{U}(\mathfrak{so}(n)) \rightarrow \mathcal{C}_n^0 \cong Q_n$, such that $\psi_n(J_{ij}) = \omega_{ij} := \frac{1}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i)$. The irreducible spin representations of $\mathfrak{so}(n)$ are realised on the simple Q_n -modules L_0 and L_2 when n is even and on L_1 when n is odd.*

Proof. As this is well known, we give merely a sketch of the argument. To show that ψ_n defines a homomorphism, it suffices to observe that the ω_{ij} satisfy the same commutation relations (5.20) as the J_{ij} , and this is straightforward. The surjectivity of ψ_n is evident from the observation that $\omega_{ij} = \gamma_i \gamma_j$, which shows that the image of ψ_n contains the whole of $\mathcal{C}_n^0 \simeq Q_n$. \square

6. THE ALGEBRA $Q_n(\ell)$ FOR $\ell > 4$.

The Definition 2.5 defines a quotient $Q_n = Q_n(\ell)$ of $\text{TL}_n(q)$ which depends on the order $\ell = |q^2|$. As we shall be considering various ℓ , we henceforth consistently denote this quotient by $Q_n(\ell)$. Thus the Ising algebra Q_n of the last section is denoted $Q_n(4)$. If we take q^2 to be $\exp(\frac{2\pi i}{\ell})$, then $q = \pm \exp(\frac{\pi i}{\ell})$, and the constant δ such that $f_j^2 = \delta f_j$ may be taken to be $\delta = \exp(\frac{\pi i}{\ell}) + \exp(-\frac{\pi i}{\ell}) = 2 \cos(\frac{\pi}{\ell})$. In this section we study the simple modules $L_0(n), L_1(n), \dots, L_{\ell-2}(n)$ of $Q_n(\ell)$. Note that $L_i(n)$ makes sense only if $n \equiv i \pmod{2}$. To illustrate the method, we begin by discussing the case $\ell = 5$.

6.1. The algebras $Q_n(5)$. An elementary study of the g function on $\mathbb{N}' = \{n \in \mathbb{N} \mid n \not\equiv -1 \pmod{\ell}\}$ leads, just as in (5.4) to the following formulae for the dimension functions ℓ_i .

$$\begin{aligned}
 \ell_0 &= \sum_{t \equiv 0 \pmod{10}} w_t - \sum_{t \equiv 8 \pmod{10}} w_t = \sum_{t \equiv 0 \pmod{10}} (w_t - w_{t+8}); \\
 \ell_1 &= \sum_{t \equiv 1 \pmod{10}} w_t - \sum_{t \equiv 7 \pmod{10}} w_t = \sum_{t \equiv 1 \pmod{10}} (w_t - w_{t+6}); \\
 \ell_2 &= \sum_{t \equiv 2 \pmod{10}} w_t - \sum_{t \equiv 6 \pmod{10}} w_t = \sum_{t \equiv 2 \pmod{10}} (w_t - w_{t+4}); \\
 \ell_3 &= \sum_{t \equiv 3 \pmod{10}} w_t - \sum_{t \equiv 5 \pmod{10}} w_t = \sum_{t \equiv 3 \pmod{10}} (w_t - w_{t+2}).
 \end{aligned}
 \tag{6.1}$$

We next make repeated use of the recurrence (5.5) to evaluate the right sides of the expressions in (6.1).

Note that

$$\begin{aligned}
 w_t(n+1) - w_{t+2}(n+1) &= w_{t-1}(n) + w_{t+1}(n) - w_{t+1}(n) - w_{t+3}(n) \\
 &= w_{t-1}(n) - w_{t+3}(n),
 \end{aligned}$$

and summing both sides over $t \equiv 3 \pmod{10}$, we obtain

$$\ell_3(n+1) = \ell_2(n) \text{ for even } n.
 \tag{6.2}$$

Similarly,

$$\begin{aligned}
 w_t(n+1) - w_{t+4}(n+1) &= w_{t-1}(n) + w_{t+1}(n) - w_{t+3}(n) - w_{t+5}(n) \\
 &= w_{t-1}(n) - w_{t+3}(n),
 \end{aligned}$$

and summing over $t \equiv 2 \pmod{10}$, we obtain

$$\ell_2(n+1) = \ell_1(n) + \ell_3(n) \text{ for odd } n.
 \tag{6.3}$$

Proceeding further along these lines yields the relations

$$\ell_1(n+1) = \ell_0(n) + \ell_2(n) \text{ for even } n.
 \tag{6.4}$$

and finally

$$\ell_0(n+1) = \ell_1(n) \text{ for odd } n.
 \tag{6.5}$$

Now let $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Combining the equations (6.2), (6.3), (6.4) and (6.5), we obtain

$$\begin{aligned}
 \begin{pmatrix} \ell_2(n+2) \\ \ell_0(n+2) \end{pmatrix} &= M \begin{pmatrix} \ell_2(n) \\ \ell_0(n) \end{pmatrix} \text{ for even } n \text{ and} \\
 \begin{pmatrix} \ell_1(n+2) \\ \ell_3(n+2) \end{pmatrix} &= M \begin{pmatrix} \ell_1(n) \\ \ell_3(n) \end{pmatrix} \text{ for odd } n.
 \end{aligned}
 \tag{6.6}$$

Now for $n = 1, 2, 3, \dots$, define integers a_n and b_n by

$$(6.7) \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} a_n + b_n & a_n \\ a_n & b_n \end{pmatrix}.$$

Remark 6.1. Note that the sequence $b_1, a_1, b_2, a_2, \dots = 1, 1, 2, 3, \dots$ is the Fibonacci sequence. Let us write this sequence F_1, F_2, \dots . Then for $n \geq 1$, $b_n = F_{2n-1}$ and $a_n = F_{2n}$. Thus we may write

$$(6.8) \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}.$$

Proposition 6.2. (1) *The dimensions of the simple $Q_n(5)$ -modules are given by the following formulae. For $n = 1, 2, 3, \dots$ write F_n for the n^{th} term in the Fibonacci sequence. Then*

$$(6.9) \quad \begin{pmatrix} \ell_1(2n+1) & \ell_2(2n) \\ \ell_3(2n+1) & \ell_0(2n) \end{pmatrix} = M^n.$$

That is, $\ell_0(2n) = b_n = F_{2n-1}$, $\ell_2(2n) = a_n = F_{2n}$, $\ell_1(2n+1) = a_n + b_n = F_{2n+1}$ and $\ell_3(2n+1) = a_n = F_{2n}$.

(2) *The dimensions of the semisimple algebras $Q_n(5)$ are given by*

$$\dim(Q_n(5)) = F_{2n-1},$$

where F_n is the n^{th} Fibonacci number.

Proof. The proof of the first statement is a simple induction on n using the recurrences above, the values for small n being easily calculated.

The second statement follows by taking the sum of the squares of the dimensions of the simple $Q_n(5)$ -modules. Specifically, we have $\dim(Q_{2n}(5)) = b_n^2 + a_n^2$. But from (6.7)

$$(6.10) \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{2n} = \begin{pmatrix} a_n + b_n & a_n \\ a_n & b_n \end{pmatrix}^2 = \begin{pmatrix} a_{2n} + b_{2n} & a_{2n} \\ a_{2n} & b_{2n} \end{pmatrix},$$

so that $a_n^2 + b_n^2 = b_{2n}$. A similar argument for $\dim(Q_{2n+1}(5))$ shows that

$$(6.11) \quad \begin{aligned} \dim(Q_{2n}(5)) &= b_{2n} \text{ and} \\ \dim(Q_{2n+1}(5)) &= a_{2n} + b_{2n}. \end{aligned}$$

In view of (6.8) and (6.10), the proof is complete. \square

6.2. The general case. We now consider $Q_n(\ell)$, where $\ell = |q^2| \geq 4$. By Theorem 3.6, the simple $Q_n(\ell)$ -modules are those $L_t = L_t(n)$ where $0 \leq t \leq \ell - 2$ and $i \equiv n \pmod{2}$. As usual, we write $\ell_i(n) = \dim(L_i(n))$ and regard ℓ_i as a function on the positive integers. We shall determine the ℓ_t recursively, as in the examples $\ell = 4, 5$ which have been described above.

Notation. The following notation will be convenient. Write $\mathcal{R} = \{0, 1, 2, \dots, \ell - 2\}$, $\overline{\mathcal{R}} = \{1, 2, 3, \dots, \ell - 1\}$ and $t \mapsto \bar{t}$ for the bijection $\mathcal{R} \rightarrow \overline{\mathcal{R}}$ given by $\bar{t} = \ell - 1 - t$.

We begin with two elementary observations concerning the g function (see Theorem 3.1).

Lemma 6.3. (1) If $i \in \mathbb{N}'$ and $i \equiv t \pmod{\ell}$, then

$$(6.12) \quad g(i) = i + 2\bar{t}.$$

(2) For any $i \in \mathbb{N}' = \{j \in \mathbb{N} \mid j \not\equiv -1 \pmod{\ell}\}$, we have

$$(6.13) \quad g^2(i) = i + 2\ell.$$

These observations lead directly to the following formula for ℓ_t .

Corollary 6.4. For $t \in \mathcal{R}$, we have

$$(6.14) \quad \ell_t = \sum_{i \in \mathbb{N}, i \equiv t \pmod{2\ell}} (w_i - w_{i+2\bar{t}}).$$

Proof. This is a direct consequence of Proposition 4.9 and Lemma 6.3. \square

We are now able to prove the key inductive relations.

Proposition 6.5. (1) If $t \in \mathcal{R}$ and $t \neq 0$ or $\ell - 2$, then for $n \equiv t - 1 \pmod{2}$, we have

$$(6.15) \quad \ell_t(n+1) = \ell_{t-1}(n) + \ell_{t+1}(n).$$

(2) For $t = 0$ and n odd, we have

$$(6.16) \quad \ell_0(n+1) = \ell_1(n).$$

(3) For $t = \ell - 2$ and $n \equiv \ell - 1 \pmod{2}$, we have

$$(6.17) \quad \ell_{\ell-2}(n+1) = \ell_{\ell-3}(n).$$

Proof. We simply apply the relation (5.5) to equation (6.14). First observe that if $i \equiv t \pmod{\ell}$, then

$$(6.18) \quad w_i(n+1) - w_{i+2\bar{t}}(n+1) = w_{i-1}(n) + w_{i+1}(n) - w_{i+2\bar{t}-1}(n) - w_{i+2\bar{t}+1}(n).$$

We shall combine the terms of the right side of (6.18) in different ways, depending on the value of t . First take t such that $0 < t < \ell - 2$. Then, recalling that $\overline{t \pm 1} = \bar{t} \mp 1$, we have

$$(6.19) \quad \begin{aligned} w_i(n+1) - w_{i+2\bar{t}}(n+1) &= (w_{i-1}(n) - w_{i+2\bar{t}+1}(n)) + (w_{i+1}(n) - w_{i+2\bar{t}-1}(n)) \\ &= (w_{i-1}(n) - w_{i-1+2\bar{t}+2}(n)) + (w_{i+1}(n) - w_{i+1+2\bar{t}-2}(n)) \\ &= (w_{i-1}(n) - w_{i-1+2\bar{t}-1}(n)) + (w_{i+1}(n) - w_{i+1+2\bar{t}+1}(n)). \end{aligned}$$

Now given (6.14), summing both sides of (6.19) over $i \equiv t \pmod{\ell}$ yields the relation (6.15).

Next take $i \equiv 0 \pmod{\ell}$, i.e. $t = 0$. Then (6.18) may be written as follows. For $i \equiv 0 \pmod{\ell}$, we have

$$(6.20) \quad \begin{aligned} w_i(n+1) - w_{i+2(\ell-1)}(n+1) &= (w_{i-1}(n) - w_{i-1+2\ell}(n)) + (w_{i+1}(n) - w_{i+1+2(\ell-2)}(n)) \\ &= (w_{i-1}(n) - w_{i-1+2\ell}(n)) + (w_{i+1}(n) - w_{i+1+2\bar{1}}(n)). \end{aligned}$$

Summing both sides of (6.20) over $i \equiv 0 \pmod{\ell}$, we see that the first summand on the right is zero since all non-zero terms cancel, while the second summand is ℓ_1 by (6.14). This proves the relation (6.16).

Finally, take $i \equiv \ell - 2 \pmod{\ell}$, i.e. $t = \ell - 2$. In this case (6.18) reads as follows.

$$(6.21) \quad \begin{aligned} w_i(n+1) - w_{i+2(\ell-2)}(n+1) &= w_{i-1}(n) + w_{i+1}(n) - (w_{i+1}(n) + w_{i+3}(n)) \\ &= (w_{i-1}(n) - w_{i-1+2\ell-3}(n)). \end{aligned}$$

Summing both sides over $i \equiv \ell - 2 \pmod{\ell}$ yields the relation (6.17) and completes the proof of the proposition. \square

Corollary 6.6. *We have the following recurrence for the dimensions $\ell_i(n)$.*

$$(6.22) \quad \begin{pmatrix} \ell_0(n+1) \\ \ell_1(n+1) \\ \vdots \\ \ell_{\ell-2}(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 0 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \ell_0(n) \\ \ell_1(n) \\ \vdots \\ \ell_{\ell-2}(n) \end{pmatrix}$$

Note that since $\ell_i(n) \neq 0$ only if $i \equiv n \pmod{2}$, to usefully apply the relation (6.22), we need to square the $(\ell - 1) \times (\ell - 1)$ matrix in that relation. The result is conveniently formulated in terms of the following matrices. Define $m \times m$ matrices as follows.

$$(6.23) \quad \begin{aligned} M_{11}(m) &= \begin{pmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 1 \end{pmatrix}, \quad M_{22}(m) = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 2 \end{pmatrix} \\ M_{12}(m) &= \begin{pmatrix} 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 2 \end{pmatrix}, \quad M_{21}(m) = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 2 & 1 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

Theorem 6.7. *Let $\ell = |q^2| \geq 4$. The dimensions $\ell_i(n)$ of the simple $Q_n(\ell)$ -modules are determined by the initial conditions $\ell_i(\ell - 1) = w_i(\ell - 1)$ for $i \equiv \ell - 1 \pmod{2}$, $\ell_i(\ell - 2) = w_i(\ell - 2)$ for $i \equiv \ell \pmod{2}$, as well as the following recurrences. Let*

$\ell \geq 6$. If ℓ is even, then

$$(6.24) \quad \mathbf{L}_{00}(n+2) := \begin{pmatrix} \ell_0(n+2) \\ \ell_2(n+2) \\ \vdots \\ \ell_{\ell-2}(n+2) \end{pmatrix} = M_{11} \left(\frac{\ell}{2} \right) \begin{pmatrix} \ell_0(n) \\ \ell_2(n) \\ \vdots \\ \ell_{\ell-2}(n) \end{pmatrix}$$

and

$$(6.25) \quad \mathbf{L}_{01}(n+2) := \begin{pmatrix} \ell_1(n+2) \\ \ell_3(n+2) \\ \vdots \\ \ell_{\ell-3}(n+2) \end{pmatrix} = M_{22} \left(\frac{\ell}{2} - 1 \right) \begin{pmatrix} \ell_1(n) \\ \ell_3(n) \\ \vdots \\ \ell_{\ell-3}(n) \end{pmatrix}.$$

If $\ell \geq 6$ is odd, then

$$(6.26) \quad \mathbf{L}_{10}(n+2) := \begin{pmatrix} \ell_0(n+2) \\ \ell_2(n+2) \\ \vdots \\ \ell_{\ell-3}(n+2) \end{pmatrix} = M_{12} \left(\frac{\ell-1}{2} \right) \begin{pmatrix} \ell_0(n) \\ \ell_2(n) \\ \vdots \\ \ell_{\ell-3}(n) \end{pmatrix}$$

and

$$(6.27) \quad \mathbf{L}_{11}(n+2) := \begin{pmatrix} \ell_1(n+2) \\ \ell_3(n+2) \\ \vdots \\ \ell_{\ell-2}(n+2) \end{pmatrix} = M_{21} \left(\frac{\ell-1}{2} \right) \begin{pmatrix} \ell_1(n) \\ \ell_3(n) \\ \vdots \\ \ell_{\ell-2}(n) \end{pmatrix}.$$

If $\ell = 4$ or 5 then as noted in the exposition above, the respective recurrence matrices are $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, (2) and $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Remark 6.8. We remark that our matrices M_{22} and M_{12} coincide with the matrices occurring in [18, §4.2]. However the context appears to be different.

To state Theorem 6.7 more explicitly, and in order to give explicit formulae for the dimension of the algebras $Q_n(\ell)$, we introduce the following notation. For $\ell \geq 6$, define column vectors \mathbf{w}_{ij} for $i, j \in \{0, 1\}$ as follows.

$$(6.28) \quad \text{For } \ell \text{ even, define } \mathbf{w}_{00} = \begin{pmatrix} w_0(\ell-2) \\ w_2(\ell-2) \\ \vdots \\ w_{\ell-2}(\ell-2) \end{pmatrix} \quad \text{and } \mathbf{w}_{01} = \begin{pmatrix} w_1(\ell-3) \\ w_3(\ell-3) \\ \vdots \\ w_{\ell-3}(\ell-3) \end{pmatrix}.$$

$$(6.29) \quad \text{For } \ell \text{ odd, define } \mathbf{w}_{10} = \begin{pmatrix} w_0(\ell-3) \\ w_2(\ell-3) \\ \vdots \\ w_{\ell-3}(\ell-3) \end{pmatrix} \quad \text{and } \mathbf{w}_{11} = \begin{pmatrix} w_1(\ell-2) \\ w_3(\ell-2) \\ \vdots \\ w_{\ell-2}(\ell-2) \end{pmatrix}.$$

Theorem 6.7 may now be stated as follows.

Corollary 6.9. *If ℓ is even, then*

$$(6.30) \quad \mathbf{L}_{00}(2n) = M_{11}^{n-\frac{\ell}{2}+1} \mathbf{w}_{00}, \text{ and}$$

$$(6.31) \quad \mathbf{L}_{01}(2n+1) = M_{22}^{n-\frac{\ell}{2}+2} \mathbf{w}_{01}.$$

If ℓ is odd, then

$$(6.32) \quad \mathbf{L}_{10}(2n) = M_{12}^{n-\frac{\ell-3}{2}} \mathbf{w}_{10}, \text{ and}$$

$$(6.33) \quad \mathbf{L}_{11}(2n+1) = M_{21}^{n-\frac{\ell-3}{2}} \mathbf{w}_{11}.$$

6.3. The dimension of $Q_n(\ell)$. Using the fact that the dimension of a semisimple \mathbb{C} -algebra is equal to the sum of the squares of the dimensions of its simple modules, we may now write the following closed formulae for the dimension of $Q_n(\ell)$, bearing in mind that the matrices M_{ij} are all symmetric..

Theorem 6.10. *If $\ell \geq 6$ is even, then*

$$\dim(Q_{2n}(\ell)) = \mathbf{w}_{00}^t M_{11}^{2n-\ell+2} \mathbf{w}_{00},$$

and

$$\dim(Q_{2n+1}(\ell)) = \mathbf{w}_{01}^t M_{22}^{2n-\ell+4} \mathbf{w}_{01}.$$

If ℓ is odd, then

$$\dim(Q_{2n}(\ell)) = \mathbf{w}_{10}^t M_{12}^{2n-\ell+3} \mathbf{w}_{10},$$

and

$$\dim(Q_{2n+1}(\ell)) = \mathbf{w}_{11}^t M_{21}^{2n-\ell+3} \mathbf{w}_{11}.$$

6.4. The case $\ell = 6$. We give the explicit solutions for the case $\ell = 6$.

Theorem 6.11. *Let $\ell = 6$. Then for $n \geq 1$,*

- (1) *For $n \geq 1$, we have $\ell_1(2n+1) = \dim(L_1(2n+1)) = \frac{3^n+1}{2}$ and $\ell_3(2n+1) = \dim(L_3(2n+1)) = \frac{3^n-1}{2}$.*
- (2) *$\dim(Q_{2n+1}(6)) = \frac{3^{2n+1}}{2}$.*
- (3) *For $n \geq 1$, we have $\ell_0(2n) = \dim(L_0(2n)) = \frac{3^{n-1}+1}{2}$, $\ell_2(2n) = \dim(L_2(2n)) = 3^{n-1}$ and $\ell_4(2n) = \dim(L_4(2n)) = \frac{3^{n-1}-1}{2}$.*
- (4) *For $n \geq 1$, we have $\dim(Q_{2n}(6)) = \frac{3^{2n-1}+1}{2}$.*

Note that the statements (2) and (4) of the theorem may be combined to state that for $n \geq 2$,

$$(6.34) \quad \dim(Q_n(6)) = \frac{3^{n-1} + 1}{2}.$$

Proof. It follows from Theorem 6.7 that the dimensions $\ell_1(3+2n)$ and $\ell_3(3+2n)$ of the relevant simple modules are given by the equation

$$(6.35) \quad \begin{pmatrix} \ell_1(3+2n) \\ \ell_3(3+2n) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

But an easy calculation shows that

$$(6.36) \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^n = \begin{pmatrix} \frac{3^n+1}{2} & \frac{3^n-1}{2} \\ \frac{3^n-1}{2} & \frac{3^n+1}{2} \end{pmatrix},$$

and the statement (1) follows.

The dimension of $Q_{2n+1}(6)$ is the sum of the squares of the dimensions of its simple modules, so that (2) follows immediately from (1).

Similarly, it also follows from Theorem 6.7 that the dimensions $\ell_i(2n+2)$ of the simple modules $L_i(2n+2)$ for i even, are given by the equation

$$(6.37) \quad \begin{pmatrix} \ell_0(2n+2) \\ \ell_2(2n+2) \\ \ell_4(2n+2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Further, another easy calculation shows that for $n \geq 1$,

$$(6.38) \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} \frac{3^{n-1}+1}{2} & 3^{n-1} & \frac{3^{n-1}-1}{2} \\ 3^{n-1} & 2 \times 3^{n-1} & 3^{n-1} \\ \frac{3^{n-1}-1}{2} & 3^{n-1} & \frac{3^{n-1}+1}{2} \end{pmatrix},$$

and substituting into (6.39), we find that for any positive integer $n \geq 0$,

$$(6.39) \quad \begin{pmatrix} \ell_0(2n+2) \\ \ell_2(2n+2) \\ \ell_4(2n+2) \end{pmatrix} = \begin{pmatrix} \frac{3^n+1}{2} \\ 3^n \\ \frac{3^n-1}{2} \end{pmatrix}.$$

The statement (3) follows immediately, and (4) follows by taking the sum of the squares of the dimensions of the three simple modules for $Q_n(6)$. \square

6.5. Some speculation. We conclude with some speculations on possible connections of our results with Virasoro algebras. Recall that the Virasoro algebra $\mathcal{L} = \oplus_{i \in \mathbb{Z}} \mathbb{C}L_i \oplus \mathbb{C}C$ has irreducible highest weight modules $L(c, h)$ with highest weight (c, h) , where $c, h \in \mathbb{C}$ are respectively the central charge and the eigenvalue of L_0 . It was conjectured by Friedan, Qiu and Schenker [3] and proved by Langlands [13] that $L(c, h)$ is unitarisable if and only if either

- (1) $c \geq 1$ and $h \geq 0$, or
- (2) there exist integers $m \geq 2$, r and s with $1 \leq s \leq r < m$ such that

$$c = 1 - \frac{6}{m(m+1)} \text{ and } h = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}.$$

This result bears a superficial resemblance to Jones' result on the range of values of the index of a subfactor as was mentioned in the preamble. Thus it might be expected that case (2) is somehow connected with our algebras $Q_n(\ell)$ for $\ell = 3, 4, 5, \dots$

Further, there are several instances in the literature (see, e.g. [4, 12, 17]) which hint at a connection between $Q_n(\ell)$ and the minimal unitary series of \mathcal{L} with central charge $c = 1 - \frac{6}{\ell(\ell-1)}$. Our work may provide some further evidence along those lines.

For $\ell = 3$, $c = 0$, and there is just one irreducible representation, viz. the trivial one. This is 'consistent' with $Q_n(3) = \mathbb{C}$. For $\ell = 4$, $c = \frac{1}{2}$. This case is the Ising model, or equivalently, the 2-state Potts model, as we have already observed.

For $\ell = 5$, which gives rise to the sequence of "Fibonacci algebras" $Q_n(5)$, $c = \frac{7}{10}$, leading to interesting combinatorics involving the Rogers-Ramanujan identities. This is the "tricritical Ising model", which has $N = 1$ supersymmetry. For $\ell = 6$, $c = \frac{4}{5}$, and we have the 3-state Potts model with \mathbb{Z}_3 -parafermionic symmetry.

We hope to return to this theme in a future work.

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